Final Exam

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CES 500
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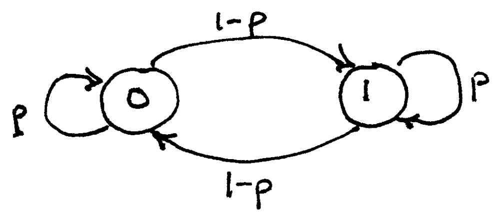
① $\mathbf{P} = \begin{vmatrix} p & 1-p \\ 1-p & p \end{vmatrix}$ <u>Show</u>: $\mathbf{P}^n = \begin{vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{vmatrix}$



Proof by induction.

Suppose $k=1$. Then $P^k = P$. Note $\begin{cases} \frac{1}{2}+\frac{1}{2}(2p-1)^1 = p \\ \frac{1}{2}-\frac{1}{2}(2p-1)^1 = 1-p \end{cases}$

With these 2 equalities,
clearly

$$P^k = \begin{vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{vmatrix} \quad \text{for } k=1.$$

Now it must be shown that for any $k \geq 1$,

$$\left(P^k = \begin{vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{vmatrix} \right) \Rightarrow \left(P^{k+1} = \begin{vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \end{vmatrix} \right)$$

Suppose the above equality is true for P^k.

Note that $P^{k+1} = \begin{vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{vmatrix} \begin{vmatrix} p & 1-p \\ 1-p & p \end{vmatrix}$

It is necessary only to show that

$$\begin{cases} \frac{1}{2}p+\frac{1}{2}p(2p-1)^k+\frac{1}{2}(1-p)-\frac{1}{2}(1-p)(2p-1)^k = \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2}p-\frac{1}{2}p(2p-1)^k+\frac{1}{2}(1-p)+\frac{1}{2}(1-p)(2p-1)^k = \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \end{cases}$$

cont'd

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① cont'd Final steps of the proof:

$$\frac{1}{2}p + \frac{1}{2}p(2p-1)^k + \frac{1}{2}(1-p) - \frac{1}{2}(1-p)(2p-1)^k$$

$$= \frac{1}{2}p + \frac{1}{2}p(2p-1)^k + \frac{1}{2} - \frac{1}{2}p - \frac{1}{2}(2p-1)^k + \frac{1}{2}p(2p-1)^k$$

$$= \frac{1}{2} - \frac{1}{2}p(2p-1)^k + p.(2p-1)^k$$

$$= \frac{1}{2} - (\frac{1}{2}-p)(2p-1)^k$$

$$= \frac{1}{2} - \frac{1}{2}(1-2p)(2p-1)^k$$

$$= \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}$$ which is a desired result.

Similarly it follows that

$$\frac{1}{2}p - \frac{1}{2}p(2p-1)^k + \frac{1}{2}(1-p) + \frac{1}{2}(1-p)(2p-1)^k = \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1}$$

the other desired result.

With these two results, the proof is complete. #

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(2) M/M/1 Let $N \equiv$ # customers in system at given time

Prove: $\text{var}(N) = \dfrac{\rho}{(1-\rho)^2}$ where $\rho = \dfrac{\lambda}{\mu} = \dfrac{\text{arrival rate}}{\text{service rate}}$

$$\text{Var}(N) = E[N^2] - (E[N])^2 .$$

Now $E[N] = \dfrac{\lambda}{\mu - \lambda}$ and $E[N^2] = \sum_{n=0}^{\infty} n^2 p_n$

$$E[N^2] = \sum_{n=0}^{\infty} n^2 \rho^n (1-\rho)$$

Consider the identity $\sum_{n=0}^{\infty} n x^n = \dfrac{x}{(1-x)^2}$

$$\frac{d}{dx}\left[1\cdot x^1 + 2\cdot x^2 + 3x^3 + \cdots\right] = \frac{(1-x^2) + 2(1-x)x}{(1-x)^4}$$

$$1 + 4x + 9x^2 + \cdots = \frac{1+x}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + \cdots = \frac{x(1-x)}{(1-x)^3}$$

Thus $E[N^2] = \sum_{n=0}^{\infty} n^2 \rho^n (1-\rho) = (1-\rho)\dfrac{\rho(1+\rho)}{(1-\rho)^2} = \dfrac{\rho(1+\rho)}{(1-\rho)^2}$

So $\text{var}(N) = \dfrac{\rho(1+\rho)}{(1-\rho)^2} - \left(\dfrac{\rho}{1-\rho}\right)^2 = \dfrac{\rho}{(1-\rho)^2}$ #

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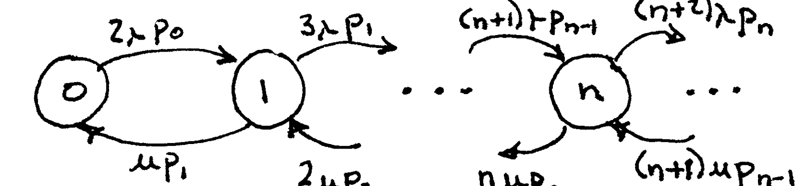
3) $\lambda_k = (k+2)\lambda \qquad k = 0, 1, 2, \cdots \qquad$ a) Find p_k

$\mu_k = k\mu \qquad k = 1, 2, 3, \cdots \qquad$ b) Find L

a)



As we have previously established,

$$p_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} p_0 \qquad \text{in general.}$$

Here, $$p_k = \frac{(2\lambda)(3\lambda)\cdots(k-1)\lambda\,(k\lambda)\,(k+1)\lambda}{\mu(2\mu)(3\mu)\cdots(k-1)\mu\,(k\mu)} p_0$$

$$p_k = \frac{(k+1)\lambda^k}{\mu^k} p_0 = (k+1)\rho^k p_0$$

$$\sum_{k=0}^{\infty} p_k = 1 = \sum_{k=0}^{\infty} (k+1)\rho^k p_0$$

$$\frac{1}{p_0} = \sum_{k=0}^{\infty} k\rho^k + \sum_{k=0}^{\infty} \rho^k = \frac{\rho}{(1-\rho)^2} + \frac{1}{1-\rho}$$

$$\frac{1}{p_0} = \frac{1}{(1-\rho)^2} \Rightarrow p_0 = (1-\rho^2)$$

Therefore $p_k = (k+1)\rho^k(1-\rho)^2 \qquad$ where $\rho = \frac{\lambda}{\mu}$

or

$$\boxed{p_k = (k+1)\left(\frac{\lambda}{\mu}\right)^k \left(1-\frac{\lambda}{\mu}\right)^2}$$

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3 b $L = \sum_{k=0}^{\infty} k\, p_i = \sum_{k=0}^{\infty} k(k+1)\rho^k (1-\rho)^2$

$$L = (1-\rho)^2 \sum_{k=0}^{\infty} (k+1)\rho^k = (1-\rho)^2 \left(\sum_{k=0} k\rho^k + \sum_{k=0} \rho^k \right)$$

$$L = (1-\rho)^2 \left(\frac{1}{(1-\rho)^2} \right) = \boxed{1}$$

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4 $\lambda_K = \begin{cases} \lambda & 0 \le K \le K \\ 2\lambda & K < k \end{cases}$

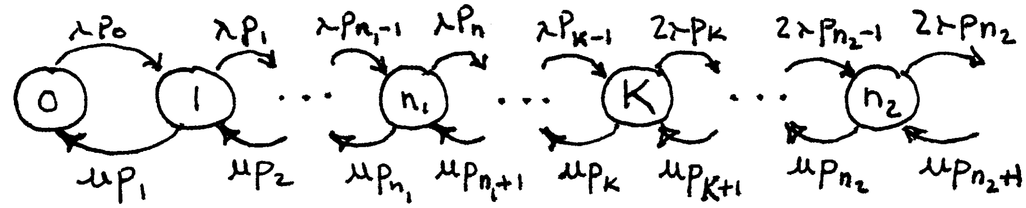
a) Find p_k

b) Requirements for stability

$\mu_K = \mu \qquad k = 1, 2, \dots$

(For convenience I will use the index 'n' rather than 'k')

a



We know that $p_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \cdot p_0$

In this case $p_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n p_0 & \text{if } 0 \le n \le K \\ \dfrac{\lambda^K (2\lambda)^{n-K}}{\mu^n} p_0 & \text{if } K < n \end{cases}$

$$p_n = \begin{cases} \rho^n \cdot p_0 & \text{if } 0 \le n \le K \\ 2^{n-K} \rho^n p_0 & \text{if } K < n \end{cases}$$

cont'd

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4 (a) cont'd if $0 \le n \le K$: $1 = \sum_{i=0}^{\infty} \rho^i p_0 \Rightarrow \frac{1}{p_0} = \frac{1}{1-\rho}$

$$\Rightarrow p_0 = 1-\rho$$

if $K < n$: $1 = \left(\sum_{i=0}^{K} \rho^i + \sum_{i=K+1}^{n} 2^{i-K} \rho^i \right) p_0$

$$\frac{1}{p_0} = \frac{1-\rho^{K+1}}{1-\rho} + 2^{-K} \sum_{i=K+1}^{n} 2^i \rho^i$$

$$\frac{1}{p_0} = \frac{1-\rho^{K+1}}{1-\rho} + 2^{-K} \sum_{j=0}^{n-K-1} 2^{j+K+1} \rho^{j+K+1}$$

$$\frac{1}{p_0} = \frac{1-\rho^{K+1}}{1-\rho} + 2^{-K} 2^{K+1} \rho^{K+1} \sum_{j=0}^{n-K-1} (2\rho)^j$$

$$\frac{1}{p_0} = \frac{1-\rho^{K+1}}{1-\rho} + 2\rho^{K+1} \left(\frac{1-\rho^{n-K}}{1-\rho} \right) = \frac{1-\rho^{K+1}+2\rho^{K+1}-2\rho^{n+1}}{1-\rho}$$

$$p_0 = \frac{1-\rho}{1-\rho^{K+1}-2\rho^{n+1}}$$

$0 \le n \le K$: $\boxed{p_n = \rho^n (1-\rho)}$

$$\boxed{p_n = \frac{2^{n-K} \rho^n (1-\rho)}{1-\rho^{K+1}-2\rho^{n+1}}}$$

if $K < n$

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4 b If $0 \le n \le K$ then $\rho < 1$ for stability.

If $K < n$, then

$$p_n = \frac{\Theta(2\rho)^n}{\Theta(\rho^n)} = \Theta(2^n)$$

$\Rightarrow$ The system is unstable.

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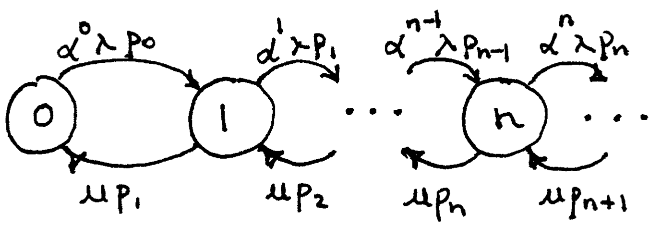
5 $\lambda_k = \alpha^k \lambda$ $\quad k \ge 0,\ 0 \le \alpha < 1$ $\quad$ a) Find p_k in terms of p_0

$\mu_k = \mu$ $\quad k \ge 1$ $\quad$ b) Find p_0



$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\mu}{\lambda} p_0 \Rightarrow p_1 = \rho^{-1} p_0 = \alpha^0 \rho^{-1} p_0$$

$$\cancel{\lambda p_0} + \mu p_2 = \cancel{\mu p_1} + \alpha \lambda p_1 = \alpha \mu p_0 \Rightarrow p_2 = \alpha^1 \rho^1 p_1 = \alpha^1 \rho^0 p_0$$

$$\alpha^1 \cancel{\lambda p_1} + \mu p_3 = \alpha^2 \lambda p_2 + \cancel{\mu p_2} \Rightarrow p_3 = \alpha^2 \rho^1 p_2 = \alpha^3 \rho^1 p_0$$

...?

a
$$p_k = \frac{(\alpha^0 \lambda)(\alpha^1 \lambda)\cdots(\alpha^{k-1}\lambda)}{\mu^k} p_0 = \rho^k \prod_{i=0}^{k-1} \alpha^i \, p_0$$

$$\boxed{p_k = \rho^k \alpha^{\frac{1}{2}k(k-1)} p_0}$$

b
$$\boxed{p_0 = \frac{1}{\sum_{k=0}^{\infty} \rho^k \alpha^{\frac{1}{2}k(k-1)}}}$$

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(6) M/M/1 <u>Show:</u> $P_r\{\text{customer spends time } x \text{ or less in queue}\}$

is given by $\begin{cases} 1-\frac{\lambda}{\mu} & \text{if } x=0 \\ 1-\frac{\lambda}{\mu}+\frac{\lambda}{\mu}\left(1-e^{-(\mu-\lambda)x}\right) & \text{if } x>0 \end{cases}$

Let $W_Q^* \equiv$ waiting time of random customer in queue

$L^* \equiv$ number of customers in system at arrival

If $x=0$ then $P\{W_Q^* \le x\} = P\{W_Q^*=0\} = p_0 = 1-\rho$ ✓

where $\rho = \frac{\lambda}{\mu}$

If $x>0$ then:

$$P\{W_Q^* \le x\} = p_0 + \sum_{n=1}^{\infty} P\{W_Q^* \le x \mid L^*=n\} \cdot P\{L^*=n\}$$

With $L^*=n$, then as soon as n customers are served, the new customer leaves the queue and enters service.

The expected wait time in queue is $\frac{n}{\mu}$.

Thus $P\{W_Q^* \le x \mid L^*=n\} \sim \Gamma(n-1, \mu)$

$$\therefore \quad P\{W_Q^* \le x\} = p_0 + \sum_{n=1}^{\infty} \int_0^x \mu e^{-\mu t} \cdot \frac{(\mu t)^{n-1}}{(n-1)!}\, dt \cdot \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$= p_0 + \rho(1-\rho) \int_0^x \mu e^{-\mu t} \underbrace{\sum_{n=1}^{\infty} \frac{(\mu\rho t)^{n-1}}{(n-1)!}}_{\sum_{m=0}^{\infty} \frac{(\mu\rho t)^m}{m!}} dt$$

cont'd

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6 cont'd

$$P\{W_Q^* \le x\} = p_0 + \rho(1-\rho)\int_0^x \mu e^{-\mu(1-\rho)t}\,dt$$

$$= p_0 + \frac{\mu\rho(1-\rho)}{-\mu(1-\rho)} \cdot e^{-\mu(1-\rho)t}\Big|_0^x$$

$$= p_0 - \rho\left(e^{-\mu(1-\rho)x} - 1\right)$$

$$P\{W_Q^* \le x\} = 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}\left(1 - e^{-(\mu-\lambda)x}\right) \qquad \#$$