Exam 2

David Bozarth
CES 500
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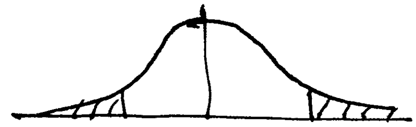
(8-5) x uniform on $\theta - 2 < x < \theta + 2$

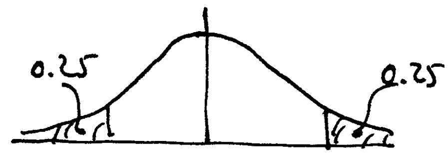
$n = 100$ $\bar{x} = 30$
$\gamma = 0.95$
Find CI of θ

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } \theta - 2 < x < \theta + 2 \\ 0 & \text{else} \end{cases}$$

Want $P\{|x - \theta| < 2\} = 0.95$



$$\text{var}(x) = E\{x^2\} - [E\{x\}]^2$$



$$E\{x^2\} = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E\{x^2\} = \frac{1}{4}\int_{28}^{32} x^2\, dx = \frac{1}{12} x^3 \Big|_{28}^{32} = \frac{1}{12}(32^3 - 28^3) = 901.33\ldots$$

$$[E\{x\}]^2 = 30^2 = 900 \Rightarrow \text{var}(x) = (901.33\ldots) - 900 = \frac{4}{3}$$

$$\theta = \bar{x} \pm z_{0.975}\frac{\sigma}{\sqrt{n}} = 30 \pm 1.967\sqrt{\frac{4}{300}}$$

$$\theta = \boxed{30 \pm 0.227}$$

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(8-9) emission rate $= X \sim \text{Poisson}\left(\frac{1}{\lambda}\right)$ $n = 200$

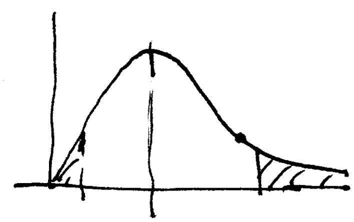
λ is mean of X Observed emission rate $= \bar{X} = \frac{2550}{200}$

$\gamma = 0.95$ Find CI of λ

Since Poisson, then $var(X) = \lambda$. But we don't know λ.

Page 311 in our text shows how to find the CI for a Poisson random variable, by using a parabola on the $\lambda - \bar{X}$ plane.

$$(\lambda - \bar{X})^2 = \frac{z^2}{n}\lambda$$



$$\lambda^2 - 2\lambda\bar{X} + \bar{X}^2 - \frac{z^2}{n}\lambda = 0$$

$$\lambda^2 - \frac{2(255)}{20}\lambda - \frac{1.967^2}{200}\lambda + \left(\frac{255}{20}\right)^2 = 0$$

$$\lambda^2 - (25.5193)\lambda + 162.5625 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{25.5193 \pm 0.9935}{2}$$

$\lambda = 12.7597 \pm 0.4967$

Note: the CI is not symmetrical about $\bar{X} = 12.75$

$$\boxed{\lambda \in [12.2629,\ 13.2564]}$$

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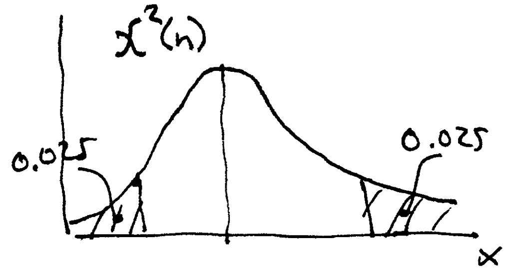
(8-31) $n = 102$ Die $k_i = 18, 15, 19, 17, 13, 20$ # times a face shows

$H_0: \sigma^2 = 0$

$\alpha = 0.05$ Estimate σ^2 by s^2

$H_1: \sigma^2 \neq 0$



$\bar{x} = 17$

For χ^2, $n = 6$

$$s^2 = \frac{1}{6}\left((18-17)^2 + (15-17)^2 + (19-17)^2 + (17-17)^2 + (13-17)^2 + (20-17)^2\right)$$

$s^2 = \frac{34}{6} = 5.66\cdots$ $\quad \chi^2(6) = 14.45$ $\quad \boxed{\text{accept } H_0}$

(8-40) $k_1 \equiv$ # successes in n trials for event A. $\quad k_2 = n - k_1$

$P(A) = p_1 = 1 - p_2$

Show: $$\frac{(k_1 - np_1)^2}{np_1} + \frac{(k_2 - np_2)^2}{np_2} = \frac{(k_1 - np_1)^2}{np_1 p_2}$$

np_1 = Expected # successes $\quad np_2$ = Expected # failures

k_1 = actual # successes $\quad k_2$ = actual # failures

cont'd

(8-40) cont'd

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$$k_2 - np_2 = n - k_1 - n(1-p_1)$$

$$\frac{(k_1-np_1)^2}{np_1p_2} - \frac{(k_1-np_1)^2}{np_1} = \frac{(k_1-np_1)^2}{np_1}\left(\frac{1}{p_2}-1\right)$$

$$= \frac{k_1^2 - 2nk_1p_1 + n^2p_1^2}{n\cancel{p_1}}\left(\frac{\cancel{1-p_2}}{p_2}\right)$$

$$= \frac{(n-k_2)^2 - 2n(n-k_2)(1-p_2) + n^2(1-p_2)^2}{np_2}$$

$$= \frac{n^2 - 2nk_2 + k_2^2 - 2n(n - k_2 - np_2 + k_2p_2) + n^2(1-2p_2+p_2^2)}{np_2}$$

$$= \frac{\cancel{n^2} - \cancel{2nk_2} + k_2^2 - \cancel{2n^2} + \cancel{2nk_2} + \cancel{2n^2p_2} - 2nk_2p_2 + \cancel{n^2} - \cancel{2n^2p_2} + n^2p_2^2}{np_2}$$

$$= \frac{(k_2-np_2)^2}{np_2} \qquad \#$$

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(9-9) Show: $\left(x(t) = c\,w(t) \text{ is WSS} \right)$ iff $\left(\begin{matrix} E\{c\} = 0 \\ \text{and} \\ w(t) = e^{j(\omega t + \theta)} \end{matrix} \right)$

($\Leftarrow$)

Suppose $E\{c\} = 0$ and $w(t) = e^{j(\omega t + \theta)}$

$$E\{c\,w(t)\} = E\{c\,e^{j(\omega t+\theta)}\} = e^{\theta} \cdot E\{c\,e^{j\omega t}\}$$

$$= e^{\theta}\left[E\{c \cos\omega t\} + j\,E\{c \sin\omega t\} \right]$$

$$E\{x(t)\} = e^{\theta}\left[0 + j0 \right]$$ is not a function of t.

$$R(t_1, t_2) = E\{c^2\, e^{j(\omega t_1 + \theta)}\, e^{-j(\omega t + \theta)}\}$$

$$= E\{c^2\, e^{j\omega(t_1 - t_2)}\} = E\{c^2\, e^{j\omega\tau}\}$$

$R(t_1, t_2)$ is not a function of t

$\Rightarrow$ $x(t)$ is WSS #

($\Rightarrow$) Suppose $x(t) = c\,w(t)$ is WSS.

Want to show that $E\{c\} = 0$ and $w(t) = e^{j(\omega t + \theta)}$

Since WSS, then $E\{c\,w(t)\} = k$ for some constant k

But $E\{|x(t)|^2\} = E\{x(t)\,x^*(t)\} = 0$ (Since WSS)

So $E\{c^2\, w(t)\, w^*(t)\} = 0$

Now $w(t)$ can not be orthogonal to itself. cont'd

Thus $E\{w(t)\,w^*(t)\} \neq 0 \Rightarrow E\{c^2\} = 0 \Rightarrow \underline{\underline{E\{c\} = 0}}$

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9.9 contd

$$R(t_1, t_2) = E\{c^2 w(t)\, w^*(t)\} = E\{c^2 w(t+\tau)\, w^*(t)\} = R(\tau)$$

is a function of $\tau = t_1 - t_2$

Therefore $w(t+\tau)\, w^*(t)$ is a function of τ.

Let $\tau = 0 \Rightarrow w(t)\, w^*(t)$ is constant because WSS.

Evidently $w(t)$ is a complex function of t,

but its conjugate multiple $|w(t)|^2$ is constant.

Then $w(t) = e^{j(\omega t + \theta)}$

#

The proof is complete.